University of California, Berkeley
Physics H7B Spring 1999 (*Strovink*)

SOLUTION TO PROBLEM SET 13

Solutions by P. Pebler

1 *Purcell 10.13*

Consider a parallel plate capacitor. The energy required to charge it to a potential difference V is $E = CV^2/2$. The capacitance increases with a dielectric to $C = \epsilon C_o = \epsilon A/4\pi s$. The potential difference is Es. Then

$$E = \frac{1}{2}CV^2 = \frac{\epsilon A E^2 s^2}{8\pi s} = \frac{\epsilon}{8\pi}E^2(As) \quad ,$$

and the energy density is

$$\epsilon \frac{E^2}{8\pi} \quad .$$

For a wave in a dielectric $B = \sqrt{\epsilon}E$ and the energy density in the magnetic field is

$$\frac{B^2}{8\pi} = \epsilon \frac{E^2}{8\pi} \quad .$$

2 *Purcell 10.16*

We use Gauss's law inside the uniform spherical charge distribution.

$$4\pi r^2 E_r = 4\pi Q_{enc} = 4\pi \frac{4\pi}{3} r^3 \rho$$

$$\mathbf{E} = \frac{4\pi}{3}\rho \mathbf{r}$$

Let the sphere of density ρ be centered at the origin, and the sphere of density $-\rho$ be centered at the location $\mathbf{s}$. The total field is

$$\mathbf{E} = \frac{4\pi}{3}\rho \mathbf{r} + \frac{4\pi}{3}(-\rho)(\mathbf{r} - \mathbf{s}) = \frac{4\pi}{3}\rho \mathbf{s} \quad .$$

In the middle of a long cylinder, we can find the field from Gauss's law.

$$2\pi r L E_r = 4\pi(\pi r^2 L \rho)$$

$$\mathbf{E} = 2\pi \rho r \hat{\mathbf{r}}$$

We are using cylindrical coordinates here so $\hat{\mathbf{r}}$ points away from the axis. The total field of two cylinders with their axes displaced by $\mathbf{s}$ is

$$\mathbf{E} = 2\pi \rho r \hat{\mathbf{r}} + 2\pi(-\rho)(r\hat{\mathbf{r}} - \mathbf{s}) = 2\pi \rho \mathbf{s} \quad .$$

3 *Purcell 11.2*

The magnetic field of a current loop with its axis on the z axis has only a z component with

$$B_z = \frac{2\pi b^2 I}{c(b^2+z^2)^{3/2}} = \frac{2m}{(b^2+z^2)^{3/2}} \quad .$$

The dipole field on this axis is all radial, which here is the z direction.

$$B'_z = B_r = \frac{2m}{r^3} = \frac{2m}{z^3}$$

So

$$B_z = \frac{z^3}{(b^2+z^2)^{3/2}} B'_z$$

and the loop field approaches the dipole field when $z \gg b$. There is a 1% difference when

$$\frac{z^3}{(b^2+z^2)^{3/2}} = 0.99 \quad ,$$

$$z = 12.2\, b \quad .$$

4 *Purcell 11.4*

The earth's radius is about 6×10^8 cm so

$$0.62\ gauss = \frac{2m}{(6 \times 10^8\ cm)^3} \quad ,$$

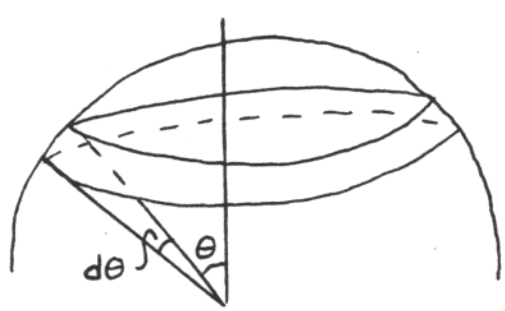
$$m = 6.7 \times 10^{25}\ erg/gauss = 6.7 \times 10^{22}\ J/T \quad .$$

If we have a current loop of radius 3×10^8 cm, we need a current I where

$$0.62\ gauss = \frac{2\pi(3 \times 10^8\ cm)^2 I}{c[(3 \times 10^8\ cm)^2 + (6 \times 10^6\ cm)^2]^{3/2}} \quad ,$$

$$I = 9.9 \times 10^{18}\ esu/s = 3.3 \times 10^9\ A \quad .$$

5 *Purcell 11.7*



We will use polar coordinates for the integration. We divide the surface into little strips subtended by the small change in polar angle $d\theta$. The surface area of one of these strips is

$$da = 2\pi(R\sin\theta)(Rd\theta) \quad .$$

The amount of charge on this strip is

$$dq = \sigma da = \frac{Q}{4\pi R^2} 2\pi R^2 \sin\theta \, d\theta = \frac{1}{2} Q \sin\theta \, d\theta \quad .$$

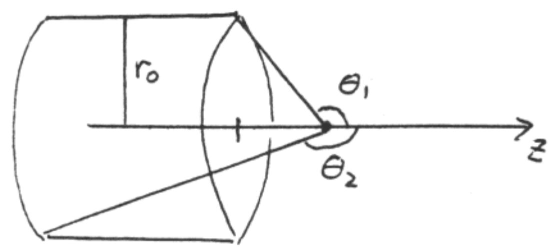
This charge revolves around with a frequency $f = \omega/2\pi$, so it represents a little current

$$dI = f \, dq = \frac{\omega Q}{4\pi} \sin\theta \, d\theta \quad .$$

Each strip contributes a moment $dm = A \, dI/c$.

$$m = \frac{1}{c} \int A \, dI = \frac{2}{c} \int_0^{\pi/2} \pi (R \sin\theta)^2 \frac{\omega Q}{4\pi} \sin\theta \, d\theta = \frac{\omega Q R^2}{2c} \int_0^{\pi/2} \sin^3\theta \, d\theta = \frac{\omega Q R^2}{3c}$$

6 *Purcell 11.9*



From Chapter 6, the field from a finite solenoid is

$$B_z = \frac{2\pi I n}{c} (\cos\theta_1 - \cos\theta_2) \quad .$$

For a semi-infinite solenoid, $\theta_2 = \pi$ and with z measuring the distance of the point outside the top of the solenoid,

$$B_z = \frac{2\pi I n}{c} \left(1 - \frac{z}{\sqrt{z^2 + r_o^2}} \right) \quad .$$

We want to maximize $B_z (dB_z/dz)$.

$$\frac{dB_z}{dz} = -\frac{2\pi I n}{c} \left(\frac{1}{\sqrt{z^2 + r_o^2}} - \frac{z^2}{(z^2 + r_o^2)^{3/2}} \right) = -\frac{2\pi I n}{c} \left(\frac{r_o^2}{(z^2 + r_o^2)^{3/2}} \right)$$

$$B_z \frac{dB_z}{dz} \propto \frac{1}{(z^2 + r_o^2)^{3/2}} \left(1 - \frac{z}{\sqrt{z^2 + r_o^2}} \right)$$

Taking a derivative and setting to zero yields the equation

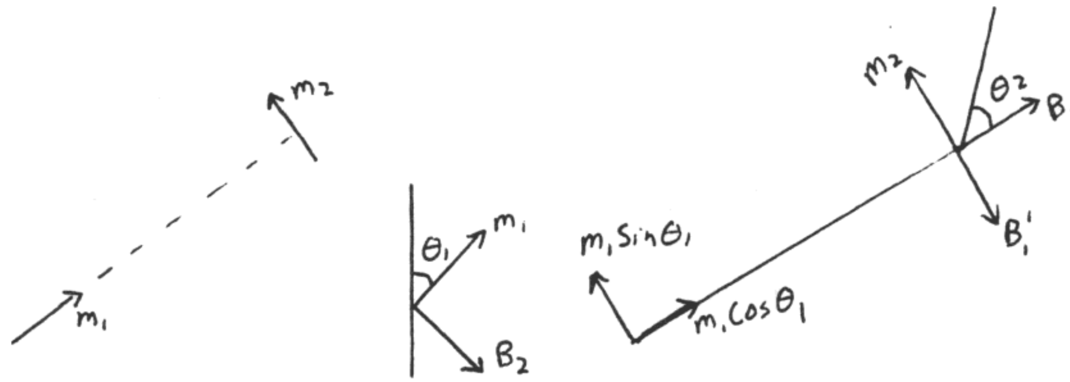
$$3z^2 - r_o^2 = 3z \sqrt{z^2 + r_o^2} \quad .$$

Squaring and solving the quadratic equation gives $z^2 = r_o^2/15$. Only the negative root solves the original equation so

$$z = -r_o \sqrt{\frac{1}{15}} \quad .$$

This is slightly inside the solenoid.

7 *Purcell 11.12*



The potential of a single dipole in a magnetic field can be chosen to be

$$U = -\mathbf{m} \cdot \mathbf{B} \quad .$$

This does not have the zero where we want our zero to be. However, for the purposes of finding work done in rotating the dipoles we may use

$$W = U_f - U_i \quad .$$

In the initial configuration, the field due to dipole 2 at m_1 is as shown above with

$$B_2 = \frac{m_2}{r^3} \quad .$$

The work required to rotate m_1 is

$$W_1 = U_f - U_i = -m_1 B_2 \cos(90 + \theta_1) - 0 = m_1 B_2 \sin\theta_1 = \frac{m_1 m_2}{r^3} \sin\theta_1 \quad .$$

To rotate m_2, we break up the field from m_1 into two parts with

$$B_1 = \frac{2m_1 \cos\theta_1}{r^3} \qquad B_1' = \frac{m_1 \sin\theta_1}{r^3} \quad .$$

The work to rotate m_2 is then

$$\begin{aligned} W_2 &= U_f - U_i = [-m_2 B_1 \cos\theta_2 - m_2 B_1' \cos(90 + \theta_2)] - [-m_2 B_1' \cos\pi] \\ &= -m_2 B_1 \cos\theta_2 + m_2 B_1' \sin\theta_2 - m_2 B_1' \\ &= -\frac{2m_1 m_2}{r^3} \cos\theta_1 \cos\theta_2 + \frac{m_1 m_2}{r^3} \sin\theta_1 \sin\theta_2 - \frac{m_1 m_2}{r^3} \sin\theta_1 \quad . \end{aligned}$$

the total work is

$$W = W_1 + W_2 = \frac{m_1 m_2}{r^3} (\sin\theta_1 \sin\theta_2 - 2\cos\theta_1 \cos\theta_2) \quad .$$

8 *Purcell 11.16*

The exterior field of a uniformly magnetized sphere turns out to be that of a magnetic dipole with dipole moment

$$m = \frac{4\pi}{3} r^3 M \quad .$$

This is something that needs to be proved, however. One can prove this by finding the field from the bound current. The bound current density is

$$\mathbf{J}_b = c\nabla \times \mathbf{M} = 0 \quad ,$$

and the bound surface current is

$$\mathbf{K}_b = c\,\mathbf{M} \times \hat{\mathbf{n}} = M \sin\theta\,\hat{\phi} \quad .$$

This is identical to the surface current of a rotating sphere with uniform surface charge. One can integrate to find the vector potential which is that of a magnetic dipole at the center. We leave this to you as an exercise.

The field at the pole is

$$B = \frac{2m}{r^3} = \frac{8\pi}{3}(750\ erg/gauss\,cm^3) = 6280\ gauss \quad .$$

At the equator

$$B = \frac{m}{r^3} = 3140\ gauss \quad .$$

To find the force, we need to know the force on a uniformly magnetized sphere in the field of a dipole. Fortunately, this is simple due to the following argument. The force on the sphere on the right must be the same if we replace the sphere on the left with a dipole at its center. This force must be equal and opposite to the force on the imaginary dipole. But the field from the sphere on the right at the dipole is that of a dipole, so the force between spheres is the same as the force between two dipoles. (This is not obvious without the argument just given.)

$$F = m_2 \left|\frac{db_{1z}}{dz}\right| = m_2 \left|\frac{d}{dz}\left(\frac{2m_1}{z^3}\right)\right| = 6\frac{m_1 m_2}{z^4} = \frac{3}{8}\frac{m_1 m_2}{r^4}$$

$$F = \frac{3}{8}\left(\frac{4\pi}{3}M\right)^2 r^2 = 3.7 \times 10^6\ dynes = 37\ N$$